

## Construction of Fuzzy $\sigma$ -Algebras Using Triangular Norms

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Generalizing the definitions given by the author [*Fuzzy Sets and Systems* 4 (1980), 83–93] we introduce and study  $T$ -fuzzy  $\sigma$ -algebras,  $T$  being any triangular norm. The main result is that for a large class of triangular norms each  $T$ -fuzzy  $\sigma$ -algebra is generated, i.e., consists of all functions  $\mu: X \rightarrow [0, 1]$  being measurable with respect to some  $\sigma$ -algebra on  $X$ .

### I. INTRODUCTION

In this paper we give an axiomatic theory of fuzzy  $\sigma$ -algebras where the operations intersection and union of fuzzy sets are assumed to be triangular norms and their duals.

We first give the definition of triangular norms as they were studied by Schweizer and Sklar [24, 25] and others in the context of statistical metric spaces. Then we derive some properties of triangular norms and give a list of examples. At the end of Section II we present an example of a nonmeasurable triangular norm.

Next we give a brief survey of papers concerning the suitability of several operators for fuzzy set theory. We mention the work done by Bellman and Giertz [4] and Hamacher [9]. In the sequel we shall follow the suggestion of Alsina *et al.* [2] and Prade [20] and perform intersection and union of fuzzy sets using triangular norms.

In Section IV we first generalize the definition of fuzzy  $\sigma$ -algebras given in [15] by replacing the operations min and max by arbitrary triangular norms and their duals. We show that if the triangular norm  $T$  is measurable and if  $(X, \mathcal{A})$  is a measurable space then the family of fuzzy sets with measurable membership functions (which were called fuzzy events by Zadeh [30]) always forms a  $T$ -fuzzy  $\sigma$ -algebra. Then, in analogy to [15] we study the relations between  $T$ -fuzzy  $\sigma$ -algebras and classical ones and present the concept of  $T$ -fuzzy measurability and the product of  $T$ -fuzzy  $\sigma$ -algebras.

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In Section V we show that for a large class of triangular norms studied by Frank [7] each  $T$ -fuzzy  $\sigma$ -algebra is generated, i.e., it consists of the family of all measurable functions  $\mu: (X, \mathcal{A}) \rightarrow ([0, 1], \mathcal{B})$ ,  $\mathcal{A}$  denoting some  $\sigma$ -algebra on  $X$  and  $\mathcal{B}$  the usual  $\sigma$ -algebra of Borel subsets of  $[0, 1]$ .

Finally we mention some problems which may be interesting to solve in the future.

## II. TRIANGULAR NORMS

Triangular norms (or briefly  $t$ -norms) have been introduced by Menger [18]. They were studied extensively by Schweizer and Sklar [24, 25], Paalman-De Miranda [19], Ling [16], Kimberling [14] and others in the context of statistical metric spaces (see Menger [18], Schweizer and Sklar [23]). They are an important tool in extending the classical triangle inequality to statistical metric spaces.

A  $t$ -norm  $T$  is defined to be a two-place function

$$T: [0, 1] \times [0, 1] \rightarrow [0, 1]$$

fulfilling the properties:

$$T(x, 1) = x \quad (\text{Boundary condition}), \quad (2.1)$$

$$T(x, y) \leq T(u, v) \quad \text{whenever } x \leq u, y \leq v \quad (\text{Monotonicity}), \quad (2.2)$$

$$T(x, y) = T(y, x) \quad (\text{Commutativity}), \quad (2.3)$$

$$T(T(x, y), z) = T(x, T(y, z)) \quad (\text{Associativity}). \quad (2.4)$$

From this definition one gets immediately

$$T(0, x) = 0. \quad (2.5)$$

From an algebraic point of view each  $t$ -norm defines a semigroup on  $[0, 1]$  with a unit 1 and an annihilator 0 and where the semigroup operation is order-preserving and commutative.

A  $t$ -norm  $T$  is said to be strict if it is continuous and strictly increasing in both places, i.e.,

$$T(x, y) < T(x, v) \quad \text{whenever } x > 0, y < v, \quad (2.6)$$

$$T(x, y) < T(u, y) \quad \text{whenever } x < u, y > 0. \quad (2.7)$$

A  $t$ -norm  $T$  is said to be Archimedean if it fulfills

$$T(x, x) < x \quad \text{for all } x \in ]0, 1[. \quad (2.8)$$

It is readily seen that each strict  $t$ -norm is Archimedean.

Given a  $t$ -norm  $T$  one can consider another two-place function

$$S: [0, 1] \times [0, 1] \rightarrow [0, 1]$$

defined by

$$S(x, y) = 1 - T(1 - x, 1 - y). \quad (2.9)$$

$S$  is called a  $t$ -conorm (or the dual of  $T$ ).

Obviously  $S$  fulfills monotonicity (2.2), commutativity (2.3), associativity (2.4) and

$$S(x, 0) = x \quad (\text{Boundary condition}). \quad (2.10)$$

We also have

$$S(x, 1) = 1.$$

Of course, if  $T$  is a strict  $t$ -norm its dual  $S$  is continuous and strictly increasing in both places, too, and if  $T$  is Archimedean then its dual  $S$  fulfills

$$S(x, x) > x \quad \text{for all } x \in ]0, 1[. \quad (2.11)$$

There are many examples of  $t$ -norms and  $t$ -conorms of which we list the most interesting ones:

$$\begin{aligned} T_s(x, y) &= \min(x, y) && \text{if } s = 0 \\ &= x \cdot y && \text{if } s = 1 \\ &= \max(x + y - 1, 0) && \text{if } s = \infty \\ &= {}^s\log \left( 1 + \frac{(s^x - 1)(s^y - 1)}{s - 1} \right) && \text{if } s \in ]0, 1[ \cup ]1, \infty[, \end{aligned} \quad (2.12)$$

$$\begin{aligned} S_s(x, y) &= \max(x, y) && \text{if } s = 0 \\ &= x + y - x \cdot y && \text{if } s = 1 \\ &= \min(x + y, 1) && \text{if } s = \infty \\ &= 1 - {}^s\log \left( 1 + \frac{(s^{1-x} - 1)(s^{1-y} - 1)}{s - 1} \right) && \text{if } s \in ]0, 1[ \cup ]1, \infty[, \end{aligned} \quad (2.13)$$

$$\begin{aligned}
 T_w(x, y) &= \min(x, y) & \text{if } \max(x, y) &= 1 \\
 &= 0 & \text{else,} & \quad (2.14)
 \end{aligned}$$

$$\begin{aligned}
 S_w(x, y) &= \max(x, y) & \text{if } \min(x, y) &= 0 \\
 &= 1 & \text{else,} & \quad (2.15)
 \end{aligned}$$

$$T^\gamma(x, y) = \frac{x \cdot y}{\gamma + (1 - \gamma)(x + y - x \cdot y)} \quad (\gamma \geq 0), \quad (2.16)$$

$$S^\gamma(x, y) = \frac{x + y + (\gamma - 2)x \cdot y}{1 + (\gamma - 1) \cdot x \cdot y} \quad (\gamma \geq 0). \quad (2.17)$$

The family of  $t$ -norms  $\{T_s | s \in [0, \infty]\}$  was studied by Frank [7], the family  $\{T^\gamma | \gamma \geq 0\}$  by Hamacher [9].

Of these  $t$ -norms  $T_0$  is not Archimedean (and hence not strict),  $T_w$  and  $T_\infty$  are Archimedean but not strict and  $T_s$ ,  $0 < s < \infty$  as well as  $T^\gamma$ ,  $\gamma \geq 0$  are strict and hence Archimedean. It is also a well-known result that  $T_0$  is the largest and  $T_w$  the smallest possible  $t$ -norm, i.e., for each  $t$ -norm  $T$  we have

$$T_w \leq T \leq T_0. \quad (2.18)$$

We also have the following limit properties (see Frank [7])

$$\lim_{s \rightarrow 0^+} T_s = T_0, \quad (2.19)$$

$$\lim_{s \rightarrow 1} T_s = T_1, \quad (2.20)$$

$$\lim_{s \rightarrow \infty} T_s = T_\infty. \quad (2.21)$$

Of course, the same holds for the family of  $t$ -conorms  $\{S_s | s \in [0, \infty]\}$ .

The following lemma will be useful for the proof of the main theorem in Section V.

**2.1. LEMMA.** *For all  $s \in [1, \infty]$  we have*

$$T_s \leq T_1. \quad (2.22)$$

*Proof.* Equation (2.22) holds obviously for  $s = 1$  and  $s = \infty$ . So it remains to be shown

$${}^s\log \left( 1 + \frac{(s^x - 1)(s^y - 1)}{s - 1} \right) \leq x \cdot y \quad (x, y \in ]0, 1[, s > 1).$$

This is equivalent to

$$\frac{s^x - 1}{s^{xy} - 1} \leq \frac{s - 1}{s^y - 1} \quad (x, y \in ]0, 1[, s > 1)$$

and, substituting  $v = s^x$ , to

$$\frac{v - 1}{v^y - 1} \leq \frac{s - 1}{s^y - 1} \quad (1 < v < s, y \in ]0, 1[).$$

So it is sufficient to show that the function

$$f_y(v) = \frac{v - 1}{v^y - 1} \quad (1 < v < s, y \in ]0, 1[)$$

is nondecreasing. Setting first  $u = v^y$  and then  $z = 1/y$  it suffices that

$$g_z(u) = \frac{u^z - 1}{u - 1} \quad (u > 1, z > 1)$$

is nondecreasing. Then we get

$$g'_z(u) = \frac{1}{(u - 1)^2} (1 + u^{z-1} [z(u - 1) - u]) \quad (u > 1, z > 1)$$

and

$$\lim_{u \rightarrow 1^+} g'_z(u) = 0 \quad (z > 1).$$

But for the function

$$h_z(u) = 1 + u^{z-1} [z(u - 1) - u] \quad (u > 1, z > 1)$$

we have

$$\lim_{u \rightarrow 1^+} h_z(u) = 0 \quad (z > 1)$$

and

$$h'_z(u) = u^{z-2} (z - 1) z(u - 1) > 0 \quad (u > 1, z > 1)$$

which implies that for any  $z > 1$ ,  $h_z$  and hence  $g_z$  are nondecreasing functions. This completes the proof. ■

If  $T$  is a  $t$ -norm and  $S$  is a  $t$ -conorm we can consider  $T(x_1, \dots, x_n)$  and  $S(x_1, \dots, x_n)$ ,  $x_1, \dots, x_n \in [0, 1]$ , defined recursively by

$$T(x_1, \dots, x_n, x_{n+1}) = T(T(x_1, \dots, x_n), x_{n+1}), \quad (2.23)$$

$$S(x_1, \dots, x_n, x_{n+1}) = S(S(x_1, \dots, x_n), x_{n+1}). \quad (2.24)$$

Due to the associativity of  $T$  and  $S$  this definition makes sense.

For any sequence  $(x_n)_{n \in \mathbb{N}} \in [0, 1]^{\mathbb{N}}$  the limits

$$\bigwedge_{n \in \mathbb{N}} x_n = \lim_{n \rightarrow \infty} T(x_1, \dots, x_n) \quad (2.25)$$

and

$$\bigvee_{n \in \mathbb{N}} x_n = \lim_{n \rightarrow \infty} S(x_1, \dots, x_n) \quad (2.26)$$

always exist because of  $T \leq T_0$  and  $S \geq S_0$ .

Note that we still have

$$S(x_1, \dots, x_n) = 1 - T(1 - x_1, \dots, 1 - x_n) \quad (2.27)$$

and

$$\bigvee_{n \in \mathbb{N}} x_n = 1 - \bigwedge_{n \in \mathbb{N}} (1 - x_n). \quad (2.28)$$

There are several methods to construct new  $t$ -norms from a given family of  $t$ -norms. We only recall the method of the so-called ordinal sums:

Let  $\{[a_i, b_i[ \mid i \in I\}$  be an at most countable family of nonempty, pairwise disjoint open subintervals of  $[0, 1]$  and  $\{T_i \mid i \in I\}$  a family of  $t$ -norms.

$$\begin{aligned} T(x, y) &= a_i + (b_i - a_i) T_i \left( \frac{x - a_i}{b_i - a_i}, \frac{y - a_i}{b_i - a_i} \right) \\ &\quad \text{if } (x, y) \in [a_i, b_i[ \times ]a_i, b_i[ \text{ for some } i \in I \\ &= \min(x, y) \quad \text{if } (x, y) \notin \bigcup_{i \in I} ([a_i, b_i[ \times ]a_i, b_i[) \end{aligned} \quad (2.29)$$

is a  $t$ -norm and called ordinal sum of  $\{([a_i, b_i[, T_i) \mid i \in I\}$ .

Obviously if  $\{S_i \mid i \in I\}$  is a family of  $t$ -conorms

$$\begin{aligned} S(x, y) &= a_i + (b_i - a_i) S_i \left( \frac{x - a_i}{b_i - a_i}, \frac{y - a_i}{b_i - a_i} \right) \\ &\quad \text{if } (x, y) \in [a_i, b_i[ \times ]a_i, b_i[ \text{ for some } i \in I \\ &= \max(x, y) \quad \text{if } (x, y) \notin \bigcup_{i \in I} ([a_i, b_i[ \times ]a_i, b_i[) \end{aligned} \quad (2.30)$$

is a  $t$ -conorm. This construction goes back to a result of Climescu [5] in the context of semigroups and was first mentioned in terms of  $t$ -norms by Schweizer and Sklar [25]. Except in trivial cases ( $I = \emptyset$  or  $I = \{i_0\}$ ,  $|a_{i_0}, b_{i_0}| = ]0, 1[$ ,  $T_{i_0}$  being a strict or Archimedean  $t$ -norm) it is readily seen that no ordinal sum can be strict or even Archimedean. But if all the  $T_i$ 's are continuous or measurable, the ordinal sum  $T$  is continuous or measurable, respectively.

The family of pairs  $\{(T_s, S_s) | s \in [0, \infty]\}$  as defined in (2.12), (2.13) and their ordinal sums have an other interesting property which will be very useful in proving the main theorem of this paper: they are the only possible pairs of a  $t$ -norm and a  $t$ -conorm  $(T, S)$  fulfilling the functional equation

$$T(x, y) + S(x, y) = x + y. \quad (2.31)$$

This was shown by Frank [7].

Although we shall work in the sequel mostly with continuous or at least measurable (with respect to Borel sets)  $t$ -norms we would like to note at the end of this section that  $t$ -norms in general need not be measurable.

2.2. EXAMPLE. Let  $A$  be a non-Borel subset of  $] \frac{3}{4}, 1[$ . Then

$$B = \{(x, y) \in [0, 1] \times [0, 1] \mid x + y = \frac{3}{2}, x \in A \text{ or } y \in A\}$$

is a non-measurable subset of  $[0, 1] \times [0, 1]$ . Hence the two-place function  $T$  on  $[0, 1] \times [0, 1]$  defined by

$$\begin{aligned} T(x, y) &= 0 && \text{if } (0 \leq x, y < 1, x + y < \tfrac{3}{2}) \text{ or } (x, y) \in B \\ &= \tfrac{1}{2} && \text{if } (0 < x, y < 1, x + y > \tfrac{3}{2}) \text{ or} \\ &&& (x + y = \tfrac{3}{2}, (x, y) \notin B) \\ &= \min(x, y) && \text{if } \max(x, y) = 1 \end{aligned} \quad (2.32)$$

is not measurable. But  $T$  is a  $t$ -norm: obviously the boundary condition, monotonicity and commutativity are fulfilled, associativity follows easily from

$$T(T(x, y), z) \neq 0 \Rightarrow \max(x, y, z) = 1.$$

### III. OPERATIONS ON FUZZY SETS

Fuzzy sets were introduced by Zadeh [29] as a generalization of Cantorian sets to be functions from a set  $X$  into the unit interval  $[0, 1]$ . The extension of the operations intersection, union and complementation in

ordinary set theory to fuzzy sets was always done pointwise: one considered two two-place functions  $T: [0, 1] \times [0, 1] \rightarrow [0, 1]$ ,  $S: [0, 1] \times [0, 1] \rightarrow [0, 1]$  and a one-place function  $N: [0, 1] \rightarrow [0, 1]$  and extended them in the usual way: if  $\mu, \nu$  are two fuzzy sets then

$$T(\mu, \nu)(x) = T(\mu(x), \nu(x)), \quad (3.1)$$

$$S(\mu, \nu)(x) = S(\mu(x), \nu(x)), \quad (3.2)$$

$$N(\mu)(x) = N(\mu(x)). \quad (3.3)$$

In this first paper Zadeh suggested to use  $T(x, y) = T_0(x, y) = \min(x, y)$  for intersection,  $S(x, y) = S_0(x, y) = \max(x, y)$  for union and  $N(x) = 1 - x$  for complementation. These operations are still used by most authors in this field.

There was also some research done in order to justify this choice of  $T$ ,  $S$  and  $N$  from an axiomatic point of view. For instance, Bellman and Giertz [4] proved that only  $T = T_0$  and  $S = S_0$  fulfill the following axioms:

$$T(x, y) = T(y, x), \quad S(x, y) = S(y, x) \quad (\text{Commutativity}), \quad (3.4)$$

$$T(T(x, y), z) = T(x, T(y, z)),$$

$$S(S(x, y), z) = S(x, S(y, z)) \quad (\text{Associativity}), \quad (3.5)$$

$$T(x, S(y, z)) = S(T(x, y), T(x, z)),$$

$$S(x, T(y, z)) = T(S(x, y), S(x, z)) \quad (\text{Distributivity}), \quad (3.6)$$

$$T(x, y) \leq T_0(x, y), \quad S(x, y) \geq S_0(x, y), \quad (3.7)$$

$$T(1, 1) = 1, \quad S(0, 0) = 0, \quad (3.8)$$

$$T \text{ and } S \text{ are continuous}, \quad (3.9)$$

$$T(x, y) \leq T(u, v) \quad \text{and} \quad S(x, y) \leq S(u, v) \quad \text{whenever} \\ x \leq u, y \leq v \quad (\text{Monotonicity}), \quad (3.10)$$

$$T(x, x) < T(y, y) \quad \text{and} \quad S(x, x) < S(y, y) \quad \text{whenever} \\ x < y. \quad (3.11)$$

Hamacher [9] showed that either axioms (3.5), (3.6), (3.7), (3.10), (3.11) or, equivalently, axioms (3.7) and (3.10) together with

$$T(x, x) = x, \quad S(x, x) = x \quad (\text{Idempotency}) \quad (3.12)$$

lead to the only possible solution  $T = T_0$  and  $S = S_0$ .



Alsina *et al.* [2] noted that if one requires the validity of the axioms (3.6), (3.10) and

$$\begin{aligned} T(x, 1) &= T(1, x) = x, \\ S(x, 0) &= S(0, x) = x \quad (\text{Boundary conditions}) \end{aligned} \quad (3.13)$$

one also obtains as only solution  $T = T_0$  and  $S = S_0$ .

Studying the proofs of these assertions one realizes that mainly distributivity (3.6) and idempotency (3.12) lead to the unique solution  $T = T_0$  and  $S = S_0$ . That means that dropping these axioms would enable us to use more general operations on fuzzy sets which behave, if the sets under consideration are crisp, precisely in the same way as intersection and union. Already Zadeh [29, 30] mentions the algebraic product (corresponding to  $T_1$ ) and the algebraic sum (corresponding basically to  $S_\infty$ ) of fuzzy sets.

This was the starting point for an extensive discussion of the suitability of several operators on fuzzy sets. Hamacher [9] gave a system of axioms implying  $T = T^\gamma$  and  $S = S^\gamma$ ,  $\gamma \geq 0$ . Lowen [17], and Trillas [28] paid special attention to the complementation of fuzzy sets. Rödder [22] and Thole *et al.* [27] presented empirical investigations of that problem. However, in Silvert [26], operations with completely different properties (symmetric sums) were considered.

The first attempts of introducing triangular norms into the theory of fuzzy sets were made by Höhle [10–12], who combined the theories of probabilistic metric spaces and fuzzy topologies in some sense. Recently (Alsina *et al.* [2], Prade [20, 21]) it was suggested to use a  $t$ -norm for intersection and its  $t$ -conorm for union of fuzzy sets. From Section II we readily see that this includes all the possible cases considered in former times (except Silvert [26]). In this paper we shall follow this idea (except the restriction that our  $t$ -norms have to be at least measurable for reasons which will be seen later).

In Dubois and Prade [6], Alsina *et al.* [2] and Prade [20] some consequences of this general setting to the algebraic structure of  $[0, 1]^X$  are studied. We only mention that in general we no longer have a lattice structure.

Choosing a general  $t$ -norm  $T$  for intersection and its dual  $S$  for union and  $N(\mu) = 1 - \mu$  for complementation means that the DeMorgan laws hold:

$$N(T(\mu, \nu)) = S(N(\mu), N(\nu)), \quad (3.14)$$

$$N(S(\mu, \nu)) = T(N(\mu), N(\nu)). \quad (3.15)$$

For the following two lemmas which will prove to be useful in the sequel we

consider a measurable space  $(X, \mathcal{A})$ , i.e., a non-empty set  $X$  and a  $\sigma$ -algebra  $\mathcal{A}$  on  $X$ . As usual, the unit interval  $[0, 1]$  is equipped with the  $\sigma$ -algebra  $\mathcal{B}$  of all Borel subsets of  $[0, 1]$ .

**3.1. LEMMA.** *Let  $T$  be a measurable  $t$ -norm,  $S$  its dual,  $\mu, \nu: (X, \mathcal{A}) \rightarrow ([0, 1], \mathcal{B})$  measurable functions and  $(\mu_n)_{n \in \mathbb{N}}$  a sequence of measurable functions from  $(X, \mathcal{A})$  into  $([0, 1], \mathcal{B})$ . Then the following functions are also measurable:*

$$T(\mu, \nu), \quad S(\mu, \nu), \quad \bigvee_{n \in \mathbb{N}} \mu_n, \quad \bigwedge_{n \in \mathbb{N}} \mu_n.$$

*Proof.* It is sufficient to show that  $T(\mu, \nu)$  is measurable. The measurability of the other functions is then an immediate consequence of the duality of  $T$  and  $S$  and of the fact that limits of measurable functions are again measurable. If  $\mu$  and  $\nu$  are measurable so is

$$\begin{aligned} (\mu, \nu): X &\rightarrow [0, 1] \times [0, 1] \\ x &\rightarrow (\mu(x), \nu(x)). \end{aligned}$$

But using

$$T(\mu, \nu) = T \circ (\mu, \nu)$$

implies that  $T(\mu, \nu)$ , being the composition of two measurable functions, is also measurable. ■

**3.2. LEMMA.** *Let  $T$  be a continuous  $t$ -norm,  $S$  its dual and  $\mu: (X, \mathcal{A}) \rightarrow ([0, 1], \mathcal{B})$  a measurable function. Then there exists a sequence  $(s_n)_{n \in \mathbb{N}}$  of measurable step functions from  $(X, \mathcal{A})$  into  $([0, 1], \mathcal{B})$  such that*

$$\mu = \bigvee_{n \in \mathbb{N}} s_n. \quad (3.16)$$

*Proof.* Let  $\mu$  be measurable and for each  $n \in \mathbb{N}$  put

$$s_n := \sum_{k=1}^{2^{n-1}} \alpha_n^k \cdot 1_{\{(2k-1)/2^n \leq \mu < 2k/2^n\}} + \alpha_n^{2^{n-1}} \cdot 1_{\{(2^n-1)/2^n \leq \mu\}},$$

where we choose  $\alpha_n^k$  such that

$$S\left(\frac{2k-2}{2^n}, \alpha_n^k\right) = \frac{2k-1}{2^n}.$$

Note first that this choice of  $\alpha_n^k$  is always possible since

$$S\left(\frac{2k-2}{2^n}, \cdot\right): [0, 1] \rightarrow \left[\frac{2k-2}{2^n}, 1\right]$$

is a surjection. Note also that each  $s_n$  is a measurable step function due to the measurability of  $\mu$ . But now it is only a matter of computation to verify

$$S(s_1, \dots, s_n) = \sum_{k=1}^{2^n-1} \frac{k-1}{2^n} \cdot 1_{\{(k-1)/2^n \leq \mu < k/2^n\}} + \frac{2^n-1}{2^n} \cdot 1_{\{(2^n-1)/2^n \leq \mu\}}$$

and hence

$$\mu = \bigvee_{n \in \mathbb{N}} s_n. \quad \blacksquare$$

However, in case the  $t$ -norm  $T$  is not continuous the assertion of Lemma 3.2 is no longer true:

**3.3. EXAMPLE.** Put  $(X, \mathcal{A}) = ([0, 1], \mathcal{B})$  and  $T = T_w$ . Suppose that a measurable function  $\mu: (X, \mathcal{A}) \rightarrow ([0, 1], \mathcal{B})$  is representable by (3.16).

Then we easily get

$$\mu(X) \subset \{1\} \cup \bigcup_{n \in \mathbb{N}} s_n(X)$$

whence it follows immediately that precisely these measurable functions can be represented by (3.16) which assume at most countably many different values in  $[0, 1]$ . But this is a proper subfamily of the family of measurable functions.

It should be mentioned that Lemmas 3.1 and 3.2 generalize a well-known characterization of measurable functions in classical measure theory: a function  $\mu: (X, \mathcal{A}) \rightarrow ([0, 1], \mathcal{B})$  is measurable if and only if there is a sequence  $(s_n)_{n \in \mathbb{N}}$  of measurable step functions from  $(X, \mathcal{A})$  into  $([0, 1], \mathcal{B})$  such that

$$\mu = \sum_{n \in \mathbb{N}} s_n.$$

#### IV. $T$ -FUZZY $\sigma$ -ALGEBRAS

In [15] we defined a subfamily  $\sigma$  of  $[0, 1]^X$  to be a fuzzy  $\sigma$ -algebra if and only if the following properties were fulfilled:

$$\alpha \text{ constant} \Rightarrow \alpha \in \sigma, \quad (4.1)$$

$$\mu \in \sigma \Rightarrow 1 - \mu \in \sigma, \quad (4.2)$$

$$(\mu_n)_{n \in \mathbb{N}} \in \sigma^{\mathbb{N}} \Rightarrow \sup_{n \in \mathbb{N}} \mu_n \in \sigma. \quad (4.3)$$

In other words, a fuzzy  $\sigma$ -algebra is a family of fuzzy sets on  $X$  containing all constant functions and being closed under complementation and countable application of the operations min and max. Since, given a classical  $\sigma$ -algebra  $\mathcal{A}$  on  $X$ , the family of all measurable functions always forms a fuzzy  $\sigma$ -algebra this definition generalizes in a straightforward manner the situation considered by Zadeh [30].

Now we shall go one step further and replace min and max by a  $t$ -norm  $T$  and its dual  $S$ , respectively.

**4.1. DEFINITION.** Let  $T$  be a  $t$ -norm and  $S$  its dual. A subfamily  $\sigma$  of  $[0, 1]^X$  is called a  $T$ -fuzzy  $\sigma$ -algebra if and only if the following properties hold:

$$\alpha \text{ constant} \Rightarrow \alpha \in \sigma, \quad (4.4)$$

$$\mu \in \sigma \Rightarrow 1 - \mu \in \sigma, \quad (4.5)$$

$$(\mu_n)_{n \in \mathbb{N}} \in \sigma^{\mathbb{N}} \Rightarrow \bigvee_{n \in \mathbb{N}} \mu_n \in \sigma. \quad (4.6)$$

The fuzzy sets in  $\sigma$  are called  $T$ -fuzzy measurable sets, the pair  $(X, \sigma)$  a  $T$ -fuzzy measurable space.

Obviously, properties (4.5), (4.6) and (2.28) imply

$$(\mu_n)_{n \in \mathbb{N}} \in \sigma^{\mathbb{N}} \Rightarrow \bigwedge_{n \in \mathbb{N}} \mu_n \in \sigma. \quad (4.7)$$

That means that a  $T$ -fuzzy  $\sigma$ -algebra is a family of fuzzy sets on  $X$  containing all constants and being closed under complementation and countable application of the operations  $T$  and  $S$ . The fuzzy  $\sigma$ -algebras studied in [15] are now  $T_0$ -fuzzy  $\sigma$ -algebras in this more general context.

Now let  $\mathcal{A}$  be again a classical  $\sigma$ -algebra on  $X$  and consider the family  $f$  all measurable functions from  $(X, \mathcal{A})$  into  $([0, 1], \mathcal{B})$ . In general, if  $T$  is an arbitrary  $t$ -norm this will not be a  $T$ -fuzzy  $\sigma$ -algebra. But we have the following result which is an immediate consequence of Lemma 3.1:

**4.2. LEMMA.** Let  $T$  be a measurable  $t$ -norm and  $\mathcal{A}$  a classical  $\sigma$ -algebra on  $X$ . Then the family of all measurable functions from  $(X, \mathcal{A})$  into  $([0, 1], \mathcal{B})$  is a  $T$ -fuzzy  $\sigma$ -algebra.

Since we want to generalize the situation in ordinary set theory and the family of all measurable functions from  $X$  into  $[0, 1]$  can be regarded as the most natural extension of a classical  $\sigma$ -algebra to the case of fuzzy sets we shall assume from now on that all  $t$ -norms considered are measurable without mentioning that explicitly. Sometimes we even shall require a  $t$ -norm to be continuous.

4.3. LEMMA. (i) Let  $(Y, \sigma)$  be a  $T$ -fuzzy measurable space and  $f: X \rightarrow Y$  a function. Then

$$f^{-1}(\sigma) = \{\mu \circ f \mid \mu \in \sigma\}$$

is a  $T$ -fuzzy  $\sigma$ -algebra on  $X$ .

(ii) Let  $(X, \sigma)$  be a  $T$ -fuzzy measurable space and  $U$  a nonempty subset of  $X$ . Then

$$\sigma/U = \{\mu/U \mid \mu \in \sigma\}$$

is a fuzzy  $\sigma$ -algebra on  $U$ .

(iii) Let  $\{\sigma_i \mid i \in I\}$  be a family of  $T$ -fuzzy  $\sigma$ -algebras on  $X$ . Then  $\bigcap_{i \in I} \sigma_i$  is a  $T$ -fuzzy  $\sigma$ -algebra on  $X$ .

*Proof.* Statement (i) is readily seen using

$$\alpha \text{ constant} \Rightarrow \alpha \circ f = \alpha,$$

$$(1 - \mu) \circ f = 1 - \mu \circ f,$$

$$\left( \bigcup_{n \in \mathbb{N}} \mu_n \right) \circ f = \bigcup_{n \in \mathbb{N}} (\mu_n \circ f).$$

Statements (ii) (being a special case of (i)) and (iii) are obvious. ■

Differently from the special case  $T = T_0$  we observe that, given a sequence  $(\mu_n)_{n \in \mathbb{N}}$  in a  $T$ -fuzzy  $\sigma$ -algebra  $\sigma$ ,  $\sup_{n \in \mathbb{N}} \mu_n$ ,  $\inf_{n \in \mathbb{N}} \mu_n$  and  $\lim_{n \rightarrow \infty} \mu_n$  need not be elements of  $\sigma$ . This can be seen from the following example.

4.4. EXAMPLE. Assume  $X = [0, 1]$ . Then

$$\begin{aligned} \sigma = & \{\alpha \in [0, 1]^X \mid \alpha \text{ constant}\} \\ & \cup \{\mu \in [0, 1]^X \mid \tfrac{1}{3} \leq \mu \leq \tfrac{2}{3}, \mu \text{ continuous}\} \end{aligned} \quad (4.8)$$

is a  $T_w$ -fuzzy  $\sigma$ -algebra on  $X$ . Now consider the sequence  $(\mu_n)_{n \in \mathbb{N}} \in \sigma^{\mathbb{N}}$  defined by

$$\begin{aligned} \mu_n: X &\rightarrow [0, 1] \\ x &\rightarrow \frac{1}{3}(nx + 1) \quad \text{if } x \leq \frac{1}{n} \\ &\rightarrow \frac{2}{3} \quad \text{if } x > \frac{1}{n}. \end{aligned}$$

Obviously

$$\lim_{n \rightarrow \infty} \mu_n = \frac{1}{3} \cdot 1_{\{0\}} + \frac{2}{3} \cdot 1_{]0, 1[}$$

is not an element of  $\sigma$ .

Next we consider, in complete analogy to [15], several functions making it easier to study  $T$ -fuzzy  $\sigma$ -algebras and their relations with classical  $\sigma$ -algebras. In the sequel we shall write  $\mathcal{P}(X)$  for the power set of  $X$ ,  $\mathcal{A}(X)$  and  $\mathcal{A}_F^T(X)$  for the family of all classical and  $T$ -fuzzy  $\sigma$ -algebras on  $X$ , respectively.

Now we introduce

$$\kappa: \mathcal{P}([0, 1]^X) \rightarrow \mathcal{A}(X),$$

where, given any family  $\varepsilon$  of fuzzy sets on  $X$ ,  $\kappa(\varepsilon)$  is defined by

$$\kappa(\varepsilon) = \sup_{\mu \in \varepsilon} \mu^{-1}(\mathcal{B}).$$

i.e., the smallest  $\sigma$ -algebra on  $X$  making each "function" in  $\varepsilon$  measurable (with respect to  $\mathcal{B}$ );

$$\zeta: \mathcal{A}(X) \rightarrow \mathcal{A}_F^T(X),$$

where, given any  $\sigma$ -algebra  $\mathcal{A}$  on  $X$ ,  $\zeta(\mathcal{A})$  denotes the family of all measurable functions from  $(X, \mathcal{A})$  into  $([0, 1], \mathcal{B})$ ;

$$\mathcal{A}: \mathcal{P}(\mathcal{P}(X)) \rightarrow \mathcal{A}(X),$$

where, given any family  $\mathcal{E}$  of crisp subsets of  $X$ ,  $\mathcal{A}(\mathcal{E})$  denotes the smallest  $\sigma$ -algebra on  $X$  containing  $\mathcal{E}$  as a subfamily;

$$\sigma_T: \mathcal{P}([0, 1]^X) \rightarrow \mathcal{A}_F^T(X),$$

where, given any family  $\varepsilon$  of fuzzy sets on  $X$ ,  $\sigma_T(\varepsilon)$  denotes the smallest

$T$ -fuzzy  $\sigma$ -algebra on  $X$  containing  $\varepsilon$  as a subfamily. Such a  $T$ -fuzzy  $\sigma$ -algebra always exists due to Lemma 4.3(iii);

$$\pi: \mathcal{P}(\mathcal{P}(X)) \rightarrow \mathcal{P}([0, 1]^X),$$

where, given any family  $\mathcal{E}$  of subsets of  $X$ ,  $\pi(\mathcal{E})$  is defined by

$$\pi(\mathcal{E}) = \{1_E \mid E \in \mathcal{E}\}$$

and

$$\lambda: \mathcal{P}([0, 1]^X) \rightarrow \mathcal{P}(\mathcal{P}(X)),$$

where, given any family  $\varepsilon$  of fuzzy sets on  $X$ ,  $\lambda(\varepsilon)$  is defined by

$$\lambda(\varepsilon) = \{\mu^{-1}(\cdot | \alpha, 1) \mid \mu \in \varepsilon, \alpha \in [0, 1]\}.$$

Note that only the functions  $\zeta$  and  $\sigma_T$  are slightly generalized in comparison with [15]. However, in the following lemma, whose proof is straightforward, we give a rather complete list of their properties.

4.5. LEMMA. (i)  $\kappa$ ,  $\mathcal{A}$ ,  $\sigma_T$  and  $\lambda$  are isotone surjections;

(ii)  $\zeta$  and  $\pi$  are isotone injections;

(iii)  $\mathcal{A}$  and  $\sigma_T$  are idempotent mappings;

$$(iv) \quad \kappa \circ \zeta = \text{id}_{\mathcal{A}(X)}; \quad (4.9)$$

$$(v) \quad \lambda \circ \pi = \text{id}_{\mathcal{P}(\mathcal{P}(X))}; \quad (4.10)$$

$$(vi) \quad \mathcal{A} \circ \lambda = \kappa \circ \sigma_T = \kappa. \quad (4.11)$$

Again in analogy to [15] we introduce the notation of a generated  $T$ -fuzzy  $\sigma$ -algebra.

4.6. DEFINITION. A  $T$ -fuzzy  $\sigma$ -algebra  $\sigma$  on  $X$  is said to be generated if and only if there exists a  $\sigma$ -algebra  $\mathcal{A}$  on  $X$  such that

$$\sigma = \zeta(\mathcal{A}). \quad (4.12)$$

In [15] it was pointed out that not every  $T_0$ -fuzzy  $\sigma$ -algebra is generated; the  $T_w$ -fuzzy  $\sigma$ -algebra  $\sigma$  defined by (4.8) is not generated either. However, as will be seen in Section V, there exist  $t$ -norms  $T$  such that each  $T$ -fuzzy  $\sigma$ -algebra is generated by some classical  $\sigma$ -algebra.

4.7. LEMMA. (i) Let  $\varepsilon$  be a family of fuzzy sets on  $X$ . Then

$$\tilde{\varepsilon} = \zeta \circ \kappa(\varepsilon) \quad (4.13)$$

is the smallest generated  $T$ -fuzzy  $\sigma$ -algebra on  $X$  containing  $\varepsilon$  as a subfamily.

(ii) A  $T$ -fuzzy  $\sigma$ -algebra  $\sigma$  is generated if and only if

$$\tilde{\sigma} = \sigma. \quad (4.14)$$

*Proof.* This is obvious.

4.8. LEMMA. Assume that the  $t$ -norm  $T$  is continuous and let  $\sigma$  be a  $T$ -fuzzy  $\sigma$ -algebra on  $X$  and  $\mathcal{A}$  a  $\sigma$ -algebra on  $X$  such that

$$\pi(\mathcal{A}) \subset \sigma. \quad (4.15)$$

Then we have

$$\zeta(\mathcal{A}) \subset \sigma.$$

*Proof.* Condition (4.15) means that all characteristic functions of elements of  $\mathcal{A}$  are contained in  $\sigma$ . This obviously implies that all measurable step functions lie in  $\sigma$ . Then our assertion is an immediate consequence of Lemma 3.2. ■

4.9. LEMMA. Assume that the  $t$ -norm  $T$  is continuous. Then we have

$$\zeta \circ \mathcal{A} = \sigma_T \circ \pi. \quad (4.16)$$

*Proof.* Using Lemma 4.8 the proof of Theorem 3.4(ii) in [15] can be copied. Note that Lemma 4.9 is a weaker form of Theorem 3.4(ii) in [15] which in general is not true if the  $t$ -norm under consideration is not continuous:

4.10. EXAMPLE. Put  $(X, \mathcal{A}) = ([0, 1], \mathcal{B})$ ,  $T = T_w$  and  $\mathcal{E} = \{[a, b] \mid a, b \in X\}$ . Then it follows from Example 3.3 that  $\sigma_T \circ \pi(\mathcal{E})$  equals the family of all measurable functions from  $(X, \mathcal{A})$  into  $([0, 1], \mathcal{B})$  assuming at most countably many different values in  $[0, 1]$  which is a proper subfamily of  $\zeta \circ \mathcal{A}(\mathcal{E})$ , the family of all measurable functions from  $(X, \mathcal{A})$  into  $([0, 1], \mathcal{B})$ .

In [15] we showed that for each function  $f: X \rightarrow Y$ , for each family  $\mathcal{E}$  of crisp subsets of  $Y$  and for each family  $\varepsilon$  of fuzzy sets on  $Y$  we have

$$\kappa(f^{-1}(\varepsilon)) = f^{-1}(\kappa(\varepsilon)), \quad (4.17)$$

$$\mathcal{A}(f^{-1}(\mathcal{E})) = f^{-1}(\mathcal{A}(\mathcal{E})), \quad (4.18)$$

$$\pi(f^{-1}(\mathcal{E})) = f^{-1}(\pi(\mathcal{E})), \quad (4.19)$$

$$\lambda(f^{-1}(\varepsilon)) = f^{-1}(\lambda(\varepsilon)). \quad (4.20)$$



4.11. LEMMA. Let  $f: X \rightarrow Y$  be a function and  $\varepsilon$  a family of fuzzy sets on  $Y$ . Then we have

$$\sigma_T(f^{-1}(\varepsilon)) = f^{-1}(\sigma_T(\varepsilon)). \quad (4.21)$$

*Proof.* Completely analogous to the proof of Proposition 3.5(iv) in [15].

4.12. LEMMA. Let  $T$  be a continuous  $t$ -norm,  $f: X \rightarrow Y$  a function,  $\mathcal{A}$  a  $\sigma$ -algebra on  $Y$  and  $\varepsilon$  a family of fuzzy sets on  $X$ . Then we have

$$(i) \quad \zeta(f^{-1}(\mathcal{A})) = f^{-1}(\zeta(\mathcal{A})), \quad (4.22)$$

$$(ii) \quad (f^{-1}(\varepsilon))^\sim = f^{-1}(\tilde{\varepsilon}). \quad (4.23)$$

*Proof.* The proofs of Proposition 3.5(vii) and (ix) in [15] can be copied word by word.

For the sake of completeness in the sequel we show that the concepts of fuzzy measurable functions and of products of fuzzy measurable spaces can be generalized to the case of  $t$ -norms. We list all necessary definitions, remarks and lemmas but we omit the proofs because it is possible to copy the corresponding proofs in [15] almost word by word.

In the following let  $(X, \xi)$  and  $(Y, \sigma)$  be two  $T$ -fuzzy measurable spaces.

4.13. DEFINITION. A function  $f: (X, \xi) \rightarrow (Y, \sigma)$  is said to be  $T$ -fuzzy measurable if and only if

$$f^{-1}(\sigma) \subset \xi. \quad (4.24)$$

4.14. REMARK. (i) Let  $f: (X, \xi) \rightarrow (Y, \rho)$  and  $g: (Y, \rho) \rightarrow (Z, \sigma)$  be two  $T$ -fuzzy measurable functions. Then the composition  $g \circ f: (X, \xi) \rightarrow (Z, \sigma)$  is  $T$ -fuzzy measurable.

(ii) Let  $f: (X, \xi) \rightarrow (Y, \sigma)$  be  $T$ -fuzzy measurable and  $U$  a nonempty subset of  $X$ . Then the restriction  $f|_U: (U, \xi|_U) \rightarrow (Y, \sigma)$  is  $T$ -fuzzy measurable.

(iii) Let  $(Y, \sigma)$  be a  $T$ -fuzzy measurable function and  $f: X \rightarrow Y$  some function. Then  $f^{-1}(\sigma)$  is the smallest  $T$ -fuzzy  $\sigma$ -algebra on  $X$  making  $f$   $T$ -fuzzy measurable.

(iv) Let  $\varepsilon$  be a family of fuzzy sets on  $Y$ . Then  $f: (X, \xi) \rightarrow (Y, \sigma(\varepsilon))$  is  $T$ -fuzzy measurable if and only if  $f^{-1}(\varepsilon) \subset \xi$ .

4.15. LEMMA. If  $f: (X, \xi) \rightarrow (Y, \sigma)$  is  $T$ -fuzzy measurable then  $f: (X, \kappa(\xi)) \rightarrow (Y, \kappa(\sigma))$  is measurable.

4.16. LEMMA. Assume that the  $t$ -norm  $T$  is continuous and let  $f: (X, \xi) \rightarrow (Y, \sigma)$  be some function. Then the following assertions are equivalent:

- (i)  $f^{-1}(\kappa(\sigma)) \subset \kappa(\xi)$ ,
- (ii)  $f^{-1}(\tilde{\sigma}) \subset \tilde{\xi}$ ,
- (iii)  $f^{-1}(\sigma) \subset \xi$ .

4.17. LEMMA. Let  $\sigma = \zeta(\mathcal{A})$  a generated  $T$ -fuzzy  $\sigma$ -algebra on  $Y$  and  $f: (X, \xi) \rightarrow (Y, \sigma)$  some function. Then  $f$  is  $T$ -fuzzy measurable if and only if

$$f^{-1}(\mathcal{A}) \subset \kappa(\xi).$$

4.18. LEMMA. Assume that the  $t$ -norm  $T$  is continuous and let  $(X, \mathcal{A})$  and  $(Y, \mathcal{S})$  be measurable spaces. Then  $f: (X, \mathcal{A}) \rightarrow (Y, \mathcal{S})$  is measurable if and only if  $f: (X, \zeta(\mathcal{A})) \rightarrow (Y, \zeta(\mathcal{S}))$  is  $T$ -fuzzy measurable.

4.19. DEFINITION. Let  $\{(X_i, \sigma_i) | i \in I\}$  be a family of  $T$ -fuzzy measurable spaces. The smallest  $T$ -fuzzy  $\sigma$ -algebra on  $\prod_{i \in I} X_i$  making each projection  $p_j: \prod_{i \in I} X_i \rightarrow X_j$ ,  $j \in I$ ,  $T$ -fuzzy measurable is called the product  $T$ -fuzzy  $\sigma$ -algebra. We denote it  $\prod_{i \in I} \sigma_i$ .

Obviously, given some  $T$ -fuzzy measurable space  $(Y, \sigma)$  and some function  $f: (Z, \sigma) \rightarrow (\prod_{i \in I} X_i, \prod_{i \in I} \sigma_i)$ ,  $f$  is  $T$ -fuzzy measurable if and only if each composition  $p_i \circ f: (Z, \sigma) \rightarrow (X_i, \sigma_i)$ ,  $i \in I$ , is  $T$ -fuzzy measurable. Moreover,  $\prod_{i \in I} \sigma_i$  is the largest  $T$ -fuzzy  $\sigma$ -algebra on  $X$  possessing this property.

4.20. LEMMA. Let  $\{(X_i, \sigma_i) | i \in I\}$  be a family of  $T$ -fuzzy measurable spaces. Then we have

$$\prod_{i \in I} \kappa(\sigma_i) = \kappa \left( \prod_{i \in I} \sigma_i \right). \quad (4.25)$$

4.21. LEMMA. Assume that the  $t$ -norm  $T$  is continuous and let  $\{(X_i, \mathcal{A}_i) | i \in I\}$  be a family of measurable spaces. Then we have

$$\prod_{i \in I} \zeta(\mathcal{A}_i) = \zeta \left( \prod_{i \in I} \mathcal{A}_i \right). \quad (4.26)$$

## V. A REPRESENTATION THEOREM

We have seen in Section IV that, for instance, there exist  $T_0$ - and  $T_w$ -fuzzy  $\sigma$ -algebras which are not generated. However, we can prove that for each  $s \in ]0, \infty]$  any  $T_s$ -fuzzy  $\sigma$ -algebra is generated.

5.1. THEOREM. *Suppose  $T = T_s$  for some  $s \in ]0, \infty]$ . Then each  $T$ -fuzzy  $\sigma$ -algebra is generated.*

*Proof.* Let  $\sigma$  be a  $T_s$ -fuzzy  $\sigma$ -algebra,  $s \in ]0, \infty]$ .

(1) In the first step we shall show that this implication holds:

$$\mu, \nu \in \sigma \Rightarrow S_\infty(\mu, \nu) \in \sigma. \quad (5.1)$$

(i) In order to do that we first choose  $\mu, \nu \in \sigma$  and then

$$\begin{aligned} \alpha_1 &= \mu, & \beta_1 &= \nu, \\ \alpha_{n+1} &= S_s(\alpha_n, \beta_n), & \beta_{n+1} &= T_s(\alpha_n, \beta_n). \end{aligned} \quad (5.2)$$

Then  $(\alpha_n)_{n \in \mathbb{N}}$  is an increasing and  $(\beta_n)_{n \in \mathbb{N}}$  a decreasing sequence in  $\sigma$ .

By induction and (2.31) we obtain for all  $n \in \mathbb{N}$

$$\alpha_n + \beta_n = \mu + \nu. \quad (5.3)$$

(ii) Next we show that for each number  $a \in [0, 1[$  there always exists a number  $c \in [0, 1[$  such that for all  $b \in [0, a]$

$$T_s(a, T_s(a, b)) \leq c \cdot T_s(a, b). \quad (5.4)$$

From Lemma 2.1 it is immediately seen that if  $s \geq 1$  we can choose  $c = a$ . If  $s < 1$  put

$$c = \frac{s^a - 1}{s - 1} < 1.$$

Then for each  $b \in [0, a]$  inequality (5.4) is equivalent to

$$\ln(1 + c^2(s^b - 1)) \geq c \cdot \ln(1 + c(s^b - 1)).$$

Expansion of the logarithms in power series leads to

$$\sum_{i=1}^{\infty} (-1)^{i-1} \frac{c^{2i}(s^b - 1)^i}{i} \geq \sum_{i=1}^{\infty} (-1)^{i-1} \frac{c^{i+1}(s^b - 1)^i}{i}$$

which is equivalent to

$$c^{i-1} \leq 1.$$

But since this latter inequality is true for all  $i \in \mathbb{N}$  (5.4) is valid for all  $s \in ]0, \infty]$ .

(iii) Next consider an element  $x \in X$  such that

$$S_{\infty}(\mu, \nu)(x) = a < 1.$$

Then because of (5.3) we have

$$\alpha_n(x) \leq a < 1 \quad (n \in \mathbb{N})$$

and using (ii) and (2.2) we obtain

$$\begin{aligned} \beta_n(x) &\leq c^{n-2} \cdot a & \text{if } s < 1 \\ &\leq a^n & \text{if } s \geq 1 \quad (n \geq 2, c \in [0, 1]). \end{aligned}$$

But this implies

$$\lim_{n \rightarrow \infty} \beta_n(x) = 0$$

and, because of (5.3),

$$\lim_{n \rightarrow \infty} \alpha_n(x) = S_{\infty}(\mu, \nu)(x).$$

(iv) If we choose an  $x \in X$  such that

$$S_{\infty}(\mu, \nu)(x) = 1$$

we obtain again

$$\lim_{n \rightarrow \infty} \alpha_n(x) = S_{\infty}(\mu, \nu)(x).$$

Assume

$$\lim_{n \rightarrow \infty} \alpha_n(x) < 1.$$

This means that there is a number  $a \in [0, 1[$  such that

$$\alpha_n(x) \leq a < 1 \quad (n \in \mathbb{N}).$$

But using analogous arguments as in (iii) this implies

$$\lim_{n \rightarrow \infty} \alpha_n(x) \geq 1$$

contradicting our assumption.

(v) If we now put

$$\begin{aligned} \gamma_1 &= \alpha_1, \\ \gamma_{n+1} &= \beta_n \quad (n \in \mathbb{N}) \end{aligned}$$

then we get

$$S_{\infty}(\mu, \nu) = \bigvee_{n \in \mathbb{N}} \gamma_n \in \sigma.$$

(2) In this step we shall show

$$\mu \in \sigma, \alpha \in [0, 1] \Rightarrow I_{\{\mu > \alpha\}} \in \sigma.$$

Choose  $\mu \in \sigma$  and  $\alpha \in [0, 1]$ . Obviously we have

$$S_{\infty}(\mu, 1 - \alpha)(x) = 1 \Leftrightarrow \mu(x) \geq \alpha \quad (x \in X).$$

Putting

$$\delta_n = S_{\infty}(\mu, 1 - \alpha) \quad (n \in \mathbb{N})$$

we obtain using (1)(ii)

$$I_{\{\mu > \alpha\}} = \bigwedge_{n \in \mathbb{N}} \delta_n$$

and hence

$$I_{\{\mu > \alpha\}} \in \sigma.$$

(3) Now it is easily seen that

$$\mathcal{A} = \{A \subset X \mid I_A \in \sigma\}$$

forms a  $\sigma$ -algebra on  $X$ . From (2) we have

$$\sigma \subset \zeta(\mathcal{A})$$

On the other hand, since each  $T_s$  is continuous we can apply Lemma 4.8 and obtain

$$\zeta(\mathcal{A}) \subset \sigma.$$

This completes the proof. ■

One possible formulation of the results of Frank [7] is the following:

The family  $\{(T_s, S_s) \mid s \in ]0, \infty[ \}$  equals the family of pairs of Archimedean  $t$ -norms and their duals fulfilling the functional equation (2.31).

So we can restate our theorem:

Let  $T$  be an Archimedean  $t$ -norm and  $S$  its dual such that equation (2.31) is fulfilled. Then each  $T$ -fuzzy  $\sigma$ -algebra is generated.

None of the assumptions can be omitted without replacing it by another supposition:

$T_w$  is Archimedean but  $(T_w, S_w)$  does not fulfill (2.31) and  $(T_0, S_0)$  fulfills (2.31) without  $T_0$  being Archimedean.

## VI. CONCLUDING REMARKS

Several problems concerning the use of  $t$ -norms in fuzzy set theory seem to be open at this moment.

It seems to be desirable not to use the whole class of  $t$ -norms but a sufficiently large class of  $t$ -norms which can be handled conveniently. For instance, as we have done in this paper, one could require continuity or at least measurability of the  $t$ -norms under consideration. On the other hand, it would be interesting to admit more general negations than  $N(x) = 1 - x$ . First attempts in this context were made by Trillas [28] and Alsina *et al.* [2].

Another natural question, in the context of fuzzy  $\sigma$ -algebras, is whether there exist more  $t$ -norms such that Theorem 5.1 holds.

Finally, in a forthcoming paper we shall use  $t$ -norms to construct and characterize fuzzy measures.

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